Week 10: BFS & Shortest Paths

Algorithms & Data Structures

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November 24, 2025

Agenda

1. Mini-Quiz
2. Assignment
3. Breadth-First Search
4. Weighted Graphs
5. Shortest Paths in DAGs
6. Dijkstra's Algorithm
7. Additional Practice

Mini-Quiz

Assignment

Feedback Assignments 8

Well done overall! Some common points:

- **Ex. 8.1 (Handshake Lemma)**: Well done! Just make sure that you first **explicitly model the situation as a graph** in order to apply graph theory. Otherwise, you are dealing with undefined quantities.
- **Ex. 8.3**
  - **Be precise**, e.g., define any specific neighbors you talk about and make your reasoning explicit.
  - **Use the terminology** introduced in the lecture. For instance, when arguing about connectedness, you can argue in terms of connected components, which have a precise definition and for which you can use all the properties shown in the lecture.
  - Even though these simpler proofs lend themselves to intuitive argument, be rigorous and **practice formalizing** them. Be **careful with intuitive arguments** (when too vague) and **proofs by example** or visual sketches for existence proofs (for counterexample a visual example is sufficient).

Remember to check the detailed feedback on Moodle! Reach out if you have any questions regarding the corrections.

Ex. 9.4

Exercise 9.4 *Bipartite graphs, Eulerian graphs and painting rooms* **(2 points)**.

In this exercise, you can use Theorem 1 above.

(a)* Prove Theorem 1.

There is a much simpler proof than shown in the master solution, which uses BFS. We'll see this later!

Ex. 9.4

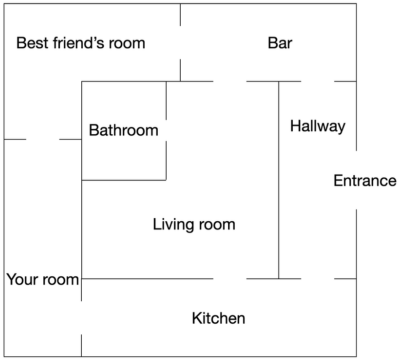
(b) Prove or disprove the following statements:

(1) Every graph G that is bipartite and Eulerian must have an even number of edges.

Ex. 9.4

(c) You recently moved in with your best friend (see floor plan below) and you would like to repaint the room walls. Every room should be painted either in red or in purple (as these are your favorite colors), and you also would like that whenever you walk from a room to another room through a door, the color changes. Is that possible?

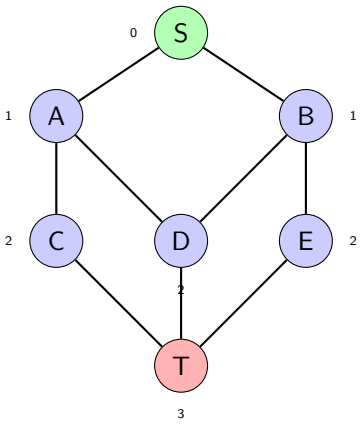
Note that there are 7 rooms (i.e. the Hallway, the Bathroom and the Kitchen are counted as rooms).



Breadth-First Search

Motivation: Finding Shortest Paths in Unweighted Graphs

Problem: Find the shortest path from a start vertex to all other vertices.



Numbers = distance from S

Key observation:

- In an **unweighted** graph, shortest path = path with **fewest edges**
- DFS doesn't guarantee shortest paths!
- We need to explore **level by level**

Solution: Breadth-First Search (BFS)

BFS: The Idea

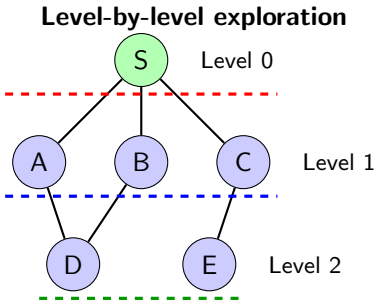
Strategy: Explore vertices in order of their distance from the start

Algorithm sketch:

1. Start at vertex s, mark distance 0
2. Explore all neighbors of s (distance 1)
3. Then explore all *their* neighbors (distance 2)
4. Continue level by level...

Data structure: Queue (FIFO)

- **Enqueue:** Add to back
- **Dequeue:** Remove from front



Key property: BFS visits vertices in order of increasing distance!

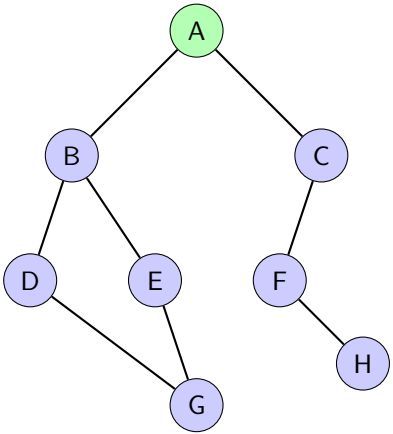
BFS: Pseudocode

```
1:  function BFS(G = (V, E), s ∈ V)
2:      Initialize dist[v] ← ∞ for all v ∈ V
3:      dist[s] ← 0
4:      Q ← {s}                                   ▷ Initialize queue with start vertex
5:      while Q ≠ ∅ do
6:          u ← dequeue(Q)
7:          for each edge (u, v) ∈ E do
8:              if v not yet visited (i.e., dist[v] = ∞) then
9:                  dist[v] ← dist[u] + 1
10:                 enqueue(Q, v)
```

Key observations

- Each vertex enters the queue **at most once**
- Vertices are dequeued in order of increasing distance
- **Runtime**: $O(|V| + |E|)$ with adjacency list (analogously to DFS)

BFS: Visual Example



Execution trace:

Step	Queue	Process
0		
1		
2		
3		
4		
5		
6		
7		
8		

Distances from A:

BFS vs DFS: Comparison

	BFS	**DFS**
Data Structure	Queue (FIFO)	Stack (LIFO) / Recursion
Exploration	Level-by-level	Deep-first, then backtrack
Shortest Paths	(unweighted)	
Space Complexity	$O(\|V\|)$	$O(\|V\|)$
Time Complexity	$O(\|V\| + \|E\|)$	$O(\|V\| + \|E\|)$
Applications	Shortest paths	Topological sort
	Connected components	Cycle detection
		Path existence

Rule of thumb: Use BFS when you care about *distance*, use DFS when you care about *structure* (cycles, connectivity, ordering).

Proof of Theorem 1

Theorem 1. A graph is bipartite if and only if it does not contain any cycle of odd length.

Proof.

Hint: Use BFS to color the vertices.

Weighted Graphs

From Unweighted to Weighted Graphs

Real-world motivation: Not all edges are equal!

Examples:

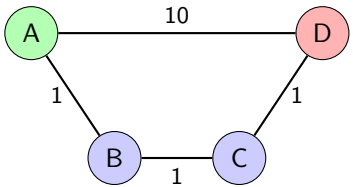
- Road networks: distances vary
- Flight routes: costs/times differ
- Networks: bandwidth varies

Definition

A **weighted graph** $G = (V, E, c)$ has a cost function:

$$c : E \to \mathbb{R}$$

assigning a cost $c(e)$ to each edge $e \in E$.



Unweighted: Shortest path is (A, D) (1 edge)
Weighted: Shortest path is (A, B, C, D) (cost 3)
Fewer edges doesn't mean lower cost!

Important: BFS no longer works! We need different algorithms.

Shortest Paths: Problem Definition

Definition: Path Cost

For a path $P = (v_0, v_1, \ldots, v_\ell)$, the **cost** is:

$$c(P) := c(v_0, v_1) + c(v_1, v_2) + \cdots + c(v_{\ell-1}, v_\ell) = \sum_{i=0}^{\ell-1} c(v_i, v_{i+1})$$

Definition: Distance

The **distance** from u to v is:

$$d(u, v) := \min\{c(P) \mid P \text{ is a path from } u \text{ to } v\}$$

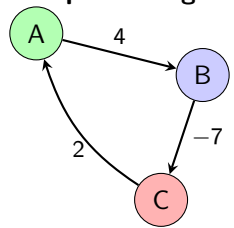
If no path exists, $d(u, v) = \infty$.

Key question: What if edge costs can be **negative**?

The Problem with Negative Edge Costs

Negative edges complicate things!

Example 1: Negative cycle

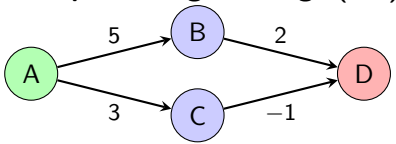


Cycle cost: $4 + (-7) + 2 = -1$

Going around the cycle decreases cost!

$\Rightarrow$ No shortest path exists

Example 2: Negative edge (OK)



Shortest path (A, D): (A, C, D) (cost 2)

This is fine—no negative cycles!

Key insight: Negative *edges* are OK, but negative *cycles* make shortest paths undefined

Assumption for Today: Non-Negative Edge Costs

For the rest of this lecture, we assume:

$$c(e) \geq 0 \quad \text{for all } e \in E$$

Why this assumption helps

- Guarantees shortest paths exist (if paths exist at all)
- Enables greedy algorithms like Dijkstra
- In many settings a reasonable assumption

What about graphs with negative edges but no negative cycles?

- Dijkstra's algorithm **fails**

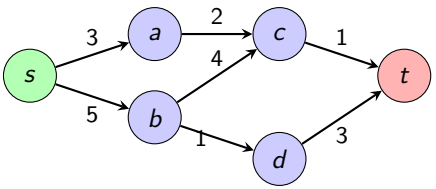
  $\Longrightarrow$ This week's lecture: Bellman-Ford

Shortest Paths in DAGs

Shortest Paths in DAGs

Recall from last week:

Question: What if our weighted graph is a DAG (Directed Acyclic Graph)?



Key advantage:

- No directed cycles $\Rightarrow$ we have a topological ordering!
- Remember: If $(v_i, v_j) \in E$, then $i < j$ in the ordering

Intuition:

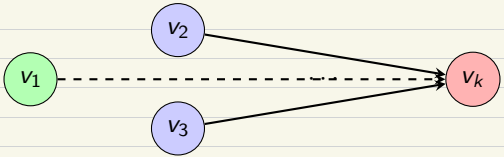
- Process vertices left-to-right
- When we process v_j, all predecessors are already done

Can we exploit this structure to find shortest paths efficiently?

Your Turn: Derive a Recurrence

Task: Suppose we have a topological ordering $v_1, v_2, \ldots, v_n$ of our DAG, with $v_1 = s$ (start vertex).

Question: How can we express $d(s, v_k)$ (shortest distance to v_k) in terms of distances to vertices that come *before* v_k in the ordering?

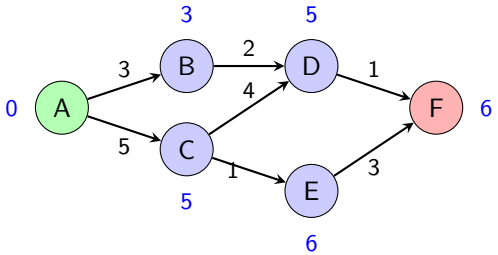


Hints:

- Any path to v_k must come from some predecessor
- In topological order, all predecessors of v_k have index $< k$
- What's the base case?

Time: 3-4 minutes

Example: DAG Shortest Paths



Topological order: A, B, C, D, E, F

Process in order:

1. A: $d[A] = 0$
2. B: $d[B] = d[A] + 3 = 3$
3. C: $d[C] = d[A] + 5 = 5$
4. D:
   - Via B: $d[B] + 2 = 5$
   - Via C: $d[C] + 4 = 9$
   - $d[D] = 5$
5. E: $d[E] = d[C] + 1 = 6$
6. F:
   - Via D: $d[D] + 1 = 6$
   - Via E: $d[E] + 3 = 9$
   - $d[F] = 6$

Result: Shortest path to F is (A, B, D, F) with cost 6

Key Insight: Why DAGs are Easy

The crucial observation:

Subproblem Ordering

In a DAG with topological ordering, we have a **natural order** to process vertices:

- When computing $d(s, v_k)$, all dependencies $d(s, v_i)$ with $i < k$ are *already computed*
- Dynamic Programming!

What we need:

1. A recurrence relation
2. An ordering that respects dependencies

In DAGs:

1. We derived the recurrence
2. Topological order gives us this for free!

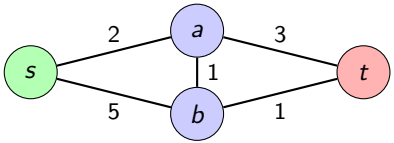
Question: What if the graph has cycles? Can we still use this approach?

Dijkstra's Algorithm

The Challenge: Undirected Graphs

So far: DAGs ⇒ topological ordering ensures all predecessors processed before each vertex

But what about undirected graphs (or directed graphs with cycles)?



No topological ordering possible

The problem:

- Every vertex can be reached from every other vertex (in cycles)
- No "natural" ordering that guarantees all predecessors are processed

What do we do?

- We need a *different* invariant
- Cannot rely on structure
- Must enforce correct processing order ourselves!

Challenge: How can we ensure we've found the shortest path to each vertex?

Dijkstra's Algorithm: The Idea

Solution: Enforce a *stronger* invariant — process vertices in order of **increasing distance**!

Dijkstra's Invariant

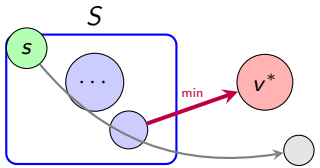
Maintain a set S of vertices for which we've determined the shortest distance.

Key property: Always add the *closest* unprocessed vertex to S next.

This ensures: $d(s, v_1) \leq d(s, v_2) \leq \cdots \leq d(s, v_n)$ for processing order.

Why this works:

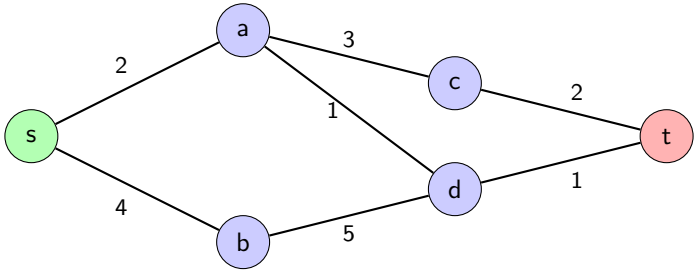
- When we add v to S with distance d
- Any other path to v must leave S at some point
- That path has length $\geq d$ (by non-negative weights and our greedy choice)
- So d is indeed the shortest distance!



Always pick closest unprocessed vertex

Your Turn: Run Dijkstra's Algorithm

Task: Run Dijkstra's algorithm on the graph below starting from s. Show the order in which vertices are added to S and the final distances.



Think about: How is this similar to the DAG approach? How is it different?

Time: 5-6 minutes

Comparison: DAG vs. Dijkstra

	DAG Algorithm	**Dijkstra's Algorithm**
Processing order	Topological order	Strictly increasing distance from s
Guarantee	All predecessors processed before each vertex	All closer vertices processed before each vertex
Why correct?	No "later" vertex can provide shortcut (DAG structure)	No "farther" vertex can provide shortcut (non-negative weights + greedy choice)
Finding order	Free! (topological sort)	Must build greedily
Recurrence	$d(s, v_k) = \min_{(v_i, v_k), i<k}\{d(s, v_i) + c(v_i, v_k)\}$	

Key insight: Both use the same recurrence, but Dijkstra needs a *stricter* processing order because it lacks the DAG structure!

Why Building the Ordering is Correct

Lemma (assuming non-negative costs)

Let S be the set of the $k-1$ closest vertices to s. Let $v^* \notin S$ be the vertex with the minimum tentative distance.

Claim: $d[v^*]$ is the shortest path to v^*, **and therefore v^* is the k-th closest vertex.**

Proof Sketch: Let P be **any** path from s to v^*.

1. Path P starts in S and ends outside, so it must cross the boundary.
2. Let (u, v) be the **first edge** in P where $u \in S$ and $v \notin S$.
3. We lower-bound the cost of P:

$$\begin{aligned} c(P) &\geq d(s,u) + c(u,v) \quad \text{(non-negative weights)} \\ &\geq d[v] \quad \text{(definition of tentative distance)} \\ &\geq d[v^*] \quad \text{(Greedy Choice: we picked } v^* \text{ over } v) \end{aligned}$$

4. **Conclusion:** No path to v^* is shorter than $d[v^*]$. Since v^* is the closest of the remaining vertices, it is the k-th closest overall. □

Correctness of The Priority Queue

Recall our recurrence:

$$d(s, v_k) = \min_{\substack{u \in S \\ u \to v_k}} \{d(s, u) + c(u, v_k)\}$$

This says: to compute shortest distance to v_k, we need to consider **all edges** from vertices in S to v_k.

In Dijkstra's algorithm, we do this efficiently using a priority queue (min-heap):

- We extract one vertex v^* at a time from the priority queue
- We only relax edges *from that one vertex*
- We never explicitly enumerate all edges from S to find the minimum

Question: How do we know that v^* is the vertex that achieves the minimum in the recurrence?
Why is extracting one vertex at a time sufficient?

Dijkstra: The Invariant

Key insight: The priority queue maintains tentative distances that already incorporate *all* edges from S seen so far.

Invariant (maintained throughout algorithm)

At any point, for each vertex $v \notin S$:

$$d[v] = \min_{\substack{u \in S \\ (u,v) \in E}} \{d[u] + c(u, v)\}$$

That is, $d[v]$ stores the **best distance to v using any edge from S**.

Why this is maintained:

1. Initially: $S = \emptyset$, $d[s] = 0$, all others $d[v] = \infty$
2. When we add vertex u to S:
   - We relax all edges (u, v): update $d[v] = \min\{d[v], d[u] + c(u, v)\}$
   - This incorporates the new edges from u into all tentative distances
3. So $d[v]$ always reflects the minimum over *all* $u \in S$

Why Extracting the Minimum is Correct

When we extract v^* with minimal $d[v^*]$:

Two crucial properties

1. $d[v^*] = \min_{u \in S, (u,v^*) \in E}\{d[u] + c(u, v^*)\}$ (by invariant) $\implies$ correct distance
2. $d[v^*] \leq d[v]$ for all $v \notin S$ (by extraction from min-heap)

What this means:

- Property (1): $d[v^*]$ already accounts for *all possible ways* to reach v^* from S
- Property (2): v^* is the closest unprocessed vertex
- Together: v^* is exactly the vertex that achieves the minimum in our recurrence!

Key point: By maintaining the invariant incrementally, the priority queue always gives us the next vertex in distance order, with its correct final distance.

Questions?

Additional Practice

Modified BFS

Exercise 10.4 *Number of minimal paths* **(1 point)**.

Let $G = (V, E)$ be an undirected graph with n vertices and m edges. Let $v, v' \in V$ be two distinct vertices and suppose that the distance between the two is k.

Describe an algorithm which counts the number of paths from v to v' of length k. The runtime of your algorithm should be at most $O(n + m)$. You are provided with the number of vertices n, and the adjacency list Adj of G. Argue why your algorithm is correct and why it satisfies the runtime bound.

Hint: *Modify BFS.*

Bonus: Longest Paths in DAGs

Exercise 9.4 (2024)

Let $G = (V, E)$ be a directed graph without directed cycles (i.e., a **directed acyclic graph** or short DAG).

Assume that $V = \{v_1, \ldots, v_n\}$ (for $n = |V| \in \mathbb{N}$) and that the sorting $v_1, v_2, \ldots, v_n$ of the vertices is a **topological sorting**.

The **goal** of this exercise is to find the **longest path** in G.

Exercise Structure

- Part a): Properties of a path in G
- Part b): DP-algorithm for finding a longest path

Part (a) - Order of a Path in G

Part (a)

Let P be a path in G. Prove that if $P = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$, then $i_1 < i_2 < \cdots < i_k$.

Part (b) - Longest-Path Finding

Part (b)

Describe a **bottom-up DP** algorithm that **returns the length of the longest path** in G in $O(|V| + |E|)$ time.

Assume that G is provided as a pair (n, Adj) of the integer $n = |V|$ and the adjacency lists Adj.

Your algorithm can **access** $\text{Adj}[u]$, which is a list of vertices to which u has a direct edge, **in constant time**. Formally, $\text{Adj}[u] := \{v \in V \mid (u, v) \in E\}$.