Week 12: Kruskal & Union-Find

Algorithms & Data Structures

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Agenda

Mini-Quiz

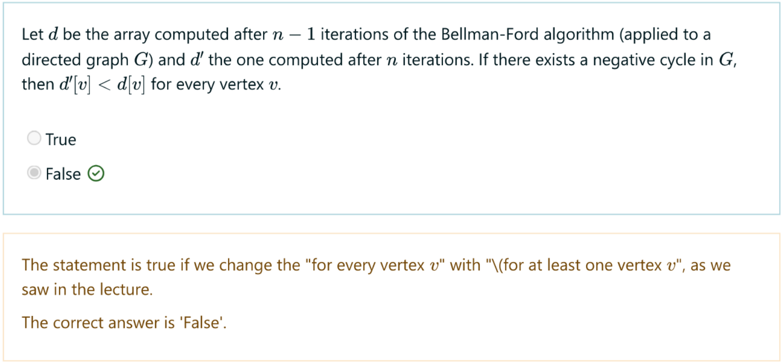
Q1

Let d be the array computed after $n-1$ iterations of the Bellman-Ford algorithm (applied to a directed graph G) and d' the one computed after n iterations. If there exists a negative cycle in G, then $d'[v] < d[v]$ for every vertex v.

True

False

Q1



Let d be the array computed after $n-1$ iterations of the Bellman-Ford algorithm (applied to a directed graph G) and d' the one computed after n iterations. If there exists a negative cycle in G, then $d'[v] < d[v]$ for every vertex v.

True

[✓ False]

The statement is true if we change the "for every vertex v" with "\(for at least one vertex v", as we saw in the lecture.

The correct answer is 'False'.

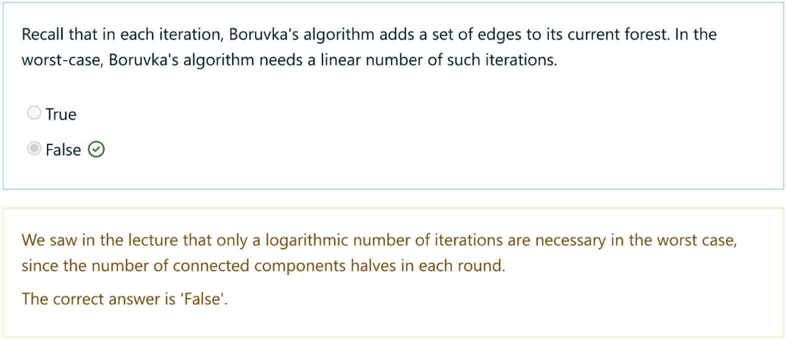
Q2

Recall that in each iteration, Boruvka's algorithm adds a set of edges to its current forest. In the worst-case, Boruvka's algorithm needs a linear number of such iterations.

True

False

Q2



Recall that in each iteration, Boruvka's algorithm adds a set of edges to its current forest. In the worst-case, Boruvka's algorithm needs a linear number of such iterations.

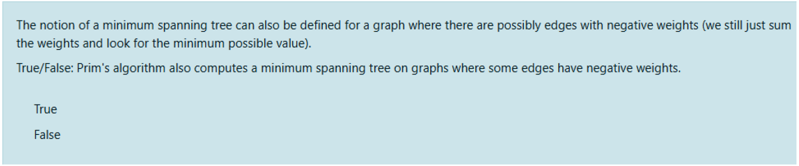
True

[✓ False] ✓

We saw in the lecture that only a logarithmic number of iterations are necessary in the worst case, since the number of connected components halves in each round.

The correct answer is 'False'.

Q3



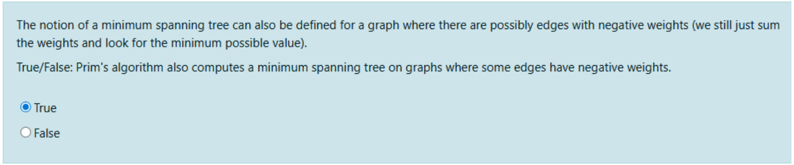
The notion of a minimum spanning tree can also be defined for a graph where there are possibly edges with negative weights (we still just sum the weights and look for the minimum possible value).

True/False: Prim's algorithm also computes a minimum spanning tree on graphs where some edges have negative weights.

True

False

Q3



The notion of a minimum spanning tree can also be defined for a graph where there are possibly edges with negative weights (we still just sum the weights and look for the minimum possible value).

True/False: Prim's algorithm also computes a minimum spanning tree on graphs where some edges have negative weights.

[✓ True]

False

Negative weights are not an issue for computing MSTs in general, because no cycles form (this is where issues show up in other algorithms). To see this even more clearly (i.e. assuming that Prim at least works for non-negative weights), assume we have a graph with some negative weights, with e.g. -100 the smallest one. We can add 100 to the weight of each edge. Now, all weights are non-negative. Moreover, any tree has its total weight increased by exactly $100(n-1)$, since each spanning tree has $n-1$ edges, so we do not change the set of MSTs.

Q4

Suppose we have stored a graph $G = (V, E)$ using adjacency lists. The runtimes of the algorithms of Prim, Boruvka and Dijkstra are all at most $O((|V| + |E|) \log |V|)$.

True

False

Q4

Suppose we have stored a graph $G = (V, E)$ using adjacency lists. The runtimes of the algorithms of Prim, Boruvka and Dijkstra are all at most $O((|V| + |E|) \log |V|)$.

[✓ True]

False

We saw this in the lecture.

The correct answer is 'True'.

Q5

Let $G = (V, E)$ be an undirected, weighted graph with positive weights. Assume that every path in this graph uses at most h edges (i.e., the length with respect to the number of edges, not the weights, is at most h). Given a source vertex v, we can compute single-source shortest paths (now "shortest" with respect to the weights as usual) in time at most $O(h|E|)$.

True

False

Q5

Let $G = (V, E)$ be an undirected, weighted graph with positive weights. Assume that every path in this graph uses at most h edges (i.e., the length with respect to the number of edges, not the weights, is at most h). Given a source vertex v, we can compute single-source shortest paths (now "shortest" with respect to the weights as usual) in time at most $O(h|E|)$.

[✓ True]

False

We can indeed modify the Bellman-Ford algorithm to only do $h-1$ "bound improvement" steps, which leads to the desired runtime. No more steps are required, since each path in the graph can only be h-long in terms of number of edges used.

The correct answer is 'True'.

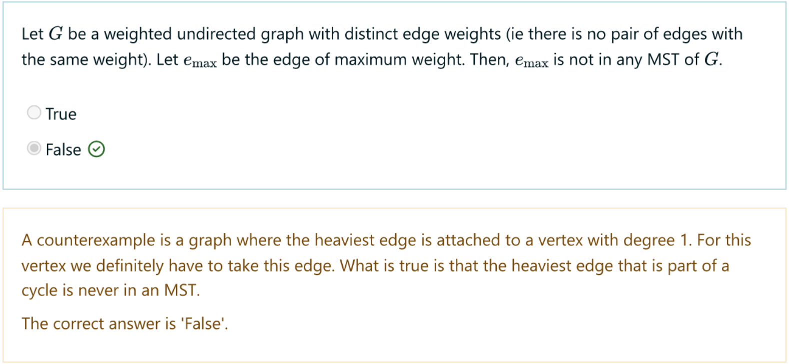
Q6

Let G be a weighted undirected graph with distinct edge weights (ie there is no pair of edges with the same weight). Let $e_{\max}$ be the edge of maximum weight. Then, $e_{\max}$ is not in any MST of G.

True

False

Q6



Let G be a weighted undirected graph with distinct edge weights (ie there is no pair of edges with the same weight). Let $e_{\max}$ be the edge of maximum weight. Then, $e_{\max}$ is not in any MST of G.

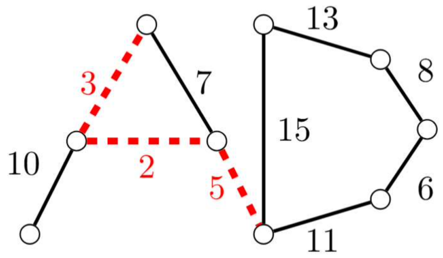
True [✓ False]

A counterexample is a graph where the heaviest edge is attached to a vertex with degree 1. For this vertex we definitely have to take this edge. What is true is that the heaviest edge that is part of a cycle is never in an MST.

The correct answer is 'False'.

Q7

We see below a graph on which we have partially executed Kruskal's algorithm. The edges already added by the algorithm are dashed and shown in red. What is the weight of the next edge added?



Q8

For the union-find data structure we saw in lecture, the worst-case runtime of a union step is at most $O(\log n)$.

True

False

Q8

For the union-find data structure we saw in lecture, the worst-case runtime of a union step is at most $O(\log n)$.

True [✓ False] ✓

We saw in lecture that union steps with even linear cost may happen, it is just that on average they take logarithmic time. So the statement is false.

The correct answer is 'False'.

Q9

As we saw from the lecture, Kruskal's algorithm uses an array repr where each vertex stores its representative. Recall that, initially, $\mathrm{repr}[i] = i$ for all i, so there are n different values stored in the array and in the end, only one value will be stored.

True or False: Suppose at some point Kruskal's algorithm has added x edges to the forest. Then, the array repr contains exactly $n - x$ distinct values.

True

False

Q9

As we saw from the lecture, Kruskal's algorithm uses an array repr where each vertex stores its representative. Recall that, initially, $\mathrm{repr}[i] = i$ for all i, so there are n different values stored in the array and in the end, only one value will be stored.

True or False: Suppose at some point Kruskal's algorithm has added x edges to the forest. Then, the array repr contains exactly $n - x$ distinct values.

[✓ True] ✓

False

The number of different representatives is equal to the number of connected components at any time. Whenever we add an edge between two different components (i.e. whenever Kruskal adds an edge), the number of connected components drops by exactly 1. We start with n components, so the statement is true.

The correct answer is 'True'.

Q10

Consider the following array of representatives in the context of finding an MST of a graph with vertices $0, 1, 2, 3, 4, 5$ using a union-find data structure:

$$[1, 1, 1, 3, 3, 5].$$

Now, suppose we unite the components of vertices 1 and 3, using the optimized union procedure which leads to $O(|V| \log |V|)$ runtime for the union steps in Kruskal's algorithm. What is the resulting array?

$[1, 1, 1, 3, 3, 5]$

$[3, 3, 3, 3, 3, 5]$

$[1, 1, 1, 1, 3, 5]$

$[1, 1, 1, 1, 1, 5]$

Q10

Consider the following array of representatives in the context of finding an MST of a graph with vertices $0, 1, 2, 3, 4, 5$ using a union-find data structure:

$$[1, 1, 1, 3, 3, 5].$$

Now, suppose we unite the components of vertices 1 and 3, using the optimized union procedure which leads to $O(|V| \log |V|)$ runtime for the union steps in Kruskal's algorithm. What is the resulting array?

- a. $[1, 1, 1, 3, 3, 5]$
- b. $[3, 3, 3, 3, 3, 5]$
- c. $[1, 1, 1, 1, 3, 5]$
- [✓ d. $[1, 1, 1, 1, 1, 5]$] ✓

Assignment

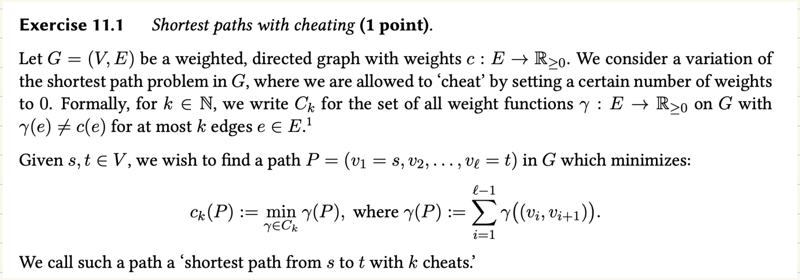
Feedback Assignment 10

Common points:

- **Ex. 10.1/2**: Well done, just be careful with the order, definitions and read the tasks carefully, making sure to answer every question.
- **Ex. 10.5**: Runtime and a correctness justification are required! In the exam, you will get deductions if you don't cover them (even if they are obvious). For these exercises, it can sometimes be easier to describe the algorithm in text to avoid mistakes in the pseudocode.

Remember to check the detailed feedback on Moodle! Reach out if you have any questions regarding the corrections.

Ex. 11.1



Exercise 11.1 *Shortest paths with cheating* **(1 point)**.

Let $G = (V, E)$ be a weighted, directed graph with weights $c : E \to \mathbb{R}_{\geq 0}$. We consider a variation of the shortest path problem in G, where we are allowed to 'cheat' by setting a certain number of weights to 0. Formally, for $k \in \mathbb{N}$, we write C_k for the set of all weight functions $\gamma : E \to \mathbb{R}_{\geq 0}$ on G with $\gamma(e) \neq c(e)$ for at most k edges $e \in E$.[1]

Given $s, t \in V$, we wish to find a path $P = (v_1 = s, v_2, \dots, v_\ell = t)$ in G which minimizes:

$$c_k(P) := \min_{\gamma \in C_k} \gamma(P), \text{ where } \gamma(P) := \sum_{i=1}^{\ell-1} \gamma\big((v_i, v_{i+1})\big).$$

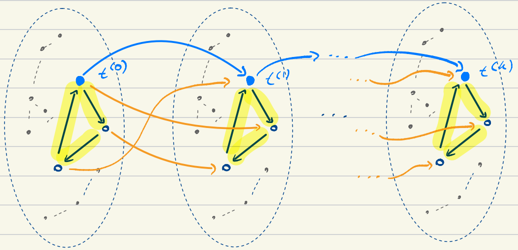
We call such a path a 'shortest path from s to t with k cheats.'

Ex. 11.1

(b) Describe an algorithm which finds the length of a shortest path from s to t with k cheats in time $O\big((k|V|)^2\big)$.

Address the following elements:

1) The graph that you use. This includes: its vertex set and edge set; whether edges are directed or not; whether vertices or edges are weighted and if so, their weight.



Solution:

We set $V' = \{v^{(\ell)} : v \in V,\ \ell \in \{0,1,2,\dots,k\}\}$, which has size $|V'| = (k+1)\cdot|V|$. We define E' and $c' : E' \to \mathbb{R}_{\geq 0}$ by:

$$\forall\,(v,w) \in E,\ 0 \leq \ell \leq k: \quad (v^{(\ell)}, w^{(\ell)}) \in E' \text{ with } c'((v^{(\ell)}, w^{(\ell)})) = c((v,w)), \tag{T1}$$

$$\forall\,(v,w) \in E,\ 0 \leq \ell \leq k-1: \quad (v^{(\ell)}, w^{(\ell+1)}) \in E' \text{ with } c'((v^{(\ell)}, w^{(\ell+1)})) = 0, \tag{T2}$$

$$\underline{\forall\, 0 \leq \ell \leq k-1:} \quad (t^{(\ell)}, t^{(\ell+1)}) \in E' \text{ with } c'((t^{(\ell)}, t^{(\ell+1)})) = 0. \tag{T3}$$

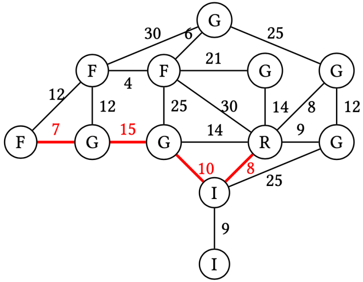
The first set of edges (T1) represents normal edge-traversal in G, the second set (T2) represents cheating, and the final set (T3) ensures that there is always a path from $s^{(0)}$ to $t^{(k)}$ in G' if there is a path from s to t in G.

Ex. 11.2

You are given a graph $G = (V, E)$ representing the towns of Switzerland. Each vertex V corresponds to a town, and there is an (undirected) edge $\{v_1, v_2\} \in E$ if and only if there exists a direct road going from town v_1 to town v_2. Additionally, there is a function $w : E \to \mathbb{N}$ such that $w(e)$ corresponds to the number of hours needed to hike over road e, and a function $\ell : V \to \{\mathrm{G}, \mathrm{F}, \mathrm{I}, \mathrm{R}\}$ that maps each town to the language that is spoken there[2]. For simplicity, we assume that only one language is spoken in each town.

Alice asks you to find an algorithm that returns the walking duration (in hours) of the shortest hike that goes through at least one town speaking each of the four languages.

For example, consider the following graph, where languages appear on vertices:

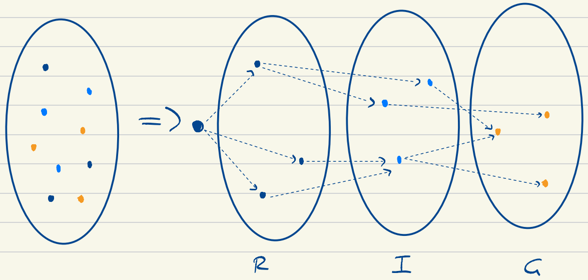


Ex. 11.2

(a) Suppose we know the order of languages encountered in the shortest hike. It first goes from an R vertex to an I vertex, then immediately to a G vertex, and reaches an F vertex in the end, after going through zero, one or more additional G vertices. In other terms, the form of the path is RIGF or RIG...GF. In this case, describe an algorithm which finds the shortest path satisfying the condition, and explain its runtime complexity. Your algorithm must have complexity at most $O((|V| + |E|) \log |V|)$.

Address the following elements:

1) The graph that you use. This includes: its vertex set and edge set; whether edges are directed or not; whether vertices or edges are weighted and if so, their weight.



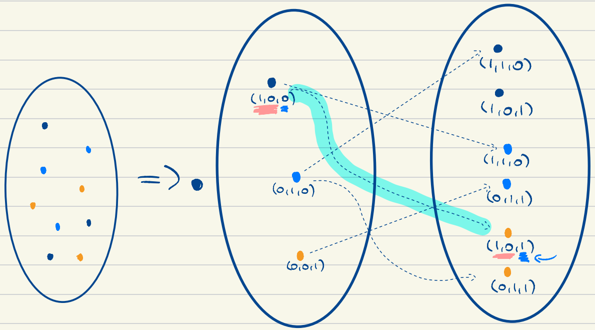
$$
\begin{aligned}
E' = \{\{v_s, (v,1)\} \mid \{u,v\} \in E, \ell(v) = \mathrm{R}\}\\
\cup \{\{(u,1),(v,2)\} \mid \{u,v\} \in E, \ell(v) = \mathrm{I}\}\\
\cup \{\{(u,2),(v,3)\} \mid \{u,v\} \in E, \ell(v) = \mathrm{G}\}\\
\cup \{\{(u,3),(v,3)\} \mid \{u,v\} \in E, \ell(v) = \mathrm{G}\}\\
\cup \{\{(u,3),(v,4)\} \mid \{u,v\} \in E, \ell(v) = \mathrm{F}\}\\
\cup \{\{(v,4), v_d\} \mid v \in V\}
\end{aligned}
$$

$$
w'(\{u',v'\}) = \begin{cases} 0 & \text{if } u' = v_s \text{ or } v' = v_d \\ w(\{u,v\}) & \text{if } u' = (u,i) \text{ and } v' = (v,j) \end{cases}
$$

Ex. 11.2

(b) Now we don't make the assumption in (a). Describe an algorithm which finds the shortest path satisfying the condition. Briefly explain your approach and the resulting runtime complexity. Your algorithm must have complexity at most $O((|V| + |E|)\log|V|)$. Address the following elements:

1) The graph that you use. This includes: its vertex set and edge set; whether edges are directed or not; whether vertices or edges are weighted and if so, their weight.

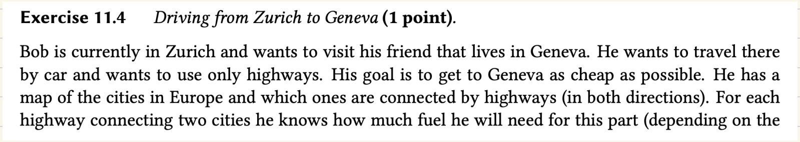


Consider the vertex set V' above, as well as the following edge set E' and weight function w':

$$
\begin{aligned}
E' = &\{\{v_s, (v, ((\ell(v) == \mathrm{G}), (\ell(v) == \mathrm{F}), (\ell(v) == \mathrm{I}), (\ell(v) == \mathrm{R})))\} \mid v \in V\} \\
&\cup \{\{(v, (1,1,1,1)), v_d\} \mid v \in V\} \\
&\cup \{\{(u, (g,f,i,r)), (v, (1, f, i, r))\} \mid (g,f,i,r) \in \{0,1\}^4, \{u,v\} \in E, \ell(v) = \mathrm{G}\} \\
&\cup \{\{(u, (g,f,i,r)), (v, (g, 1, i, r))\} \mid (g,f,i,r) \in \{0,1\}^4, \{u,v\} \in E, \ell(v) = \mathrm{F}\} \\
&\cup \{\{(u, (g,f,i,r)), (v, (g, f, 1, r))\} \mid (g,f,i,r) \in \{0,1\}^4, \{u,v\} \in E, \ell(v) = \mathrm{I}\} \\
&\cup \{\{(u, (g,f,i,r)), (v, (g, f, i, 1))\} \mid (g,f,i,r) \in \{0,1\}^4, \{u,v\} \in E, \ell(v) = \mathrm{R}\}
\end{aligned}
$$

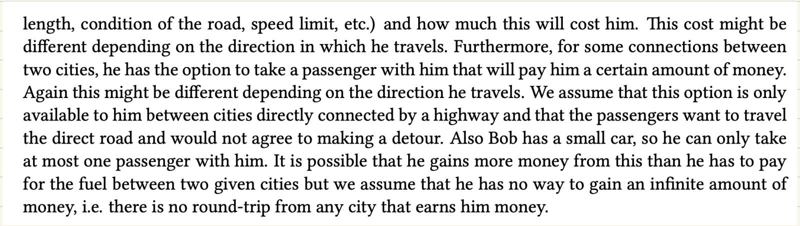
$$
w'(\{u',v'\}) = \begin{cases} 0 & \text{if } u' = v_s \text{ or } v' = v_d \\ w(\{u,v\}) & \text{if } u' = (u,(g,f,i,r)) \text{ and } v' = (v,(g,f,i,r)) \end{cases}
$$

Ex. 11.4



Exercise 11.4 *Driving from Zurich to Geneva* **(1 point)**.

Bob is currently in Zurich and wants to visit his friend that lives in Geneva. He wants to travel there by car and wants to use only highways. His goal is to get to Geneva as cheap as possible. He has a map of the cities in Europe and which ones are connected by highways (in both directions). For each highway connecting two cities he knows how much fuel he will need for this part (depending on the



length, condition of the road, speed limit, etc.) and how much this will cost him. This cost might be different depending on the direction in which he travels. Furthermore, for some connections between two cities, he has the option to take a passenger with him that will pay him a certain amount of money. Again this might be different depending on the direction he travels. We assume that this option is only available to him between cities directly connected by a highway and that the passengers want to travel the direct road and would not agree to making a detour. Also Bob has a small car, so he can only take at most one passenger with him. It is possible that he gains more money from this than he has to pay for the fuel between two given cities but we assume that he has no way to gain an infinite amount of money, i.e. there is no round-trip from any city that earns him money.

Ex. 11.4

(a) Model the problem as a graph problem.

Address the following elements:

1) The graph that you use. This includes: its vertex set and edge set; whether edges are directed or not; whether vertices or edges are weighted and if so, their weight.

Solution:

The graph $G = (V, E, w)$ is defined as follows: V is the set of cities on his map of Europe. There are directed edges between the cities (in both directions) if they are connected by a highway. The weight of any edge is the difference of the cost he needs to pay for the fuel for this highway and the money that a passenger would pay him (if available, otherwise it is just the cost he needs pay).

2) The algorithm that you apply to this graph. You can use the algorithms covered in the lecture material as subroutines, and you can use their running time bounds without proof.

Solution:

We apply Bellman-Ford starting at v_{Zurich} corresponding to Zurich since edge weights are possibly negative.

Kruskal's Algorithm

Recap: MST Algorithms So Far

What we've seen:

Prim's Algorithm:

- Grow a single tree from a starting vertex
- Always add the cheapest edge leaving the current tree
- Uses: Priority Queue
- Runtime: $O((|V| + |E|) \log |V|)$

Boruvka's Algorithm:

- Grow all components simultaneously
- Each component adds its cheapest outgoing edge
- Uses: Component tracking
- Runtime: $O((|V| + |E|) \log |V|)$

Different approach: What if we process edges in a **global** order and only pick edges that are safe?

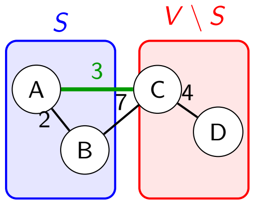
Kruskal's Algorithm: The Idea

New Approach: Instead of growing trees, look at min-weight edges and ensure that no cycles are introduced!

Kruskal's Greedy Strategy

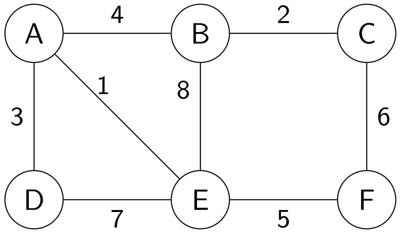
1. Sort all edges by weight (cheapest first)
2. Go through edges in order
3. For each edge (u, v):
   - If adding it doesn't create a cycle $\rightarrow$ **add it to MST**
   - Otherwise $\rightarrow$ **skip it**
4. Stop when we have $|V| - 1$ edges

- **Recall: Minimum Cut Property** states that any minimum weight edge that crosses a cut must be in some MST.



Kruskal: Manual Walkthrough

Min-Cut Property gives us a greedy choice. How does Kruskal use it to select the edges?



Sorted Edges:

Edge	**Weight**
{A, E}	1
{B, C}	2
{D, A}	3
{A, B}	4
{E, F}	5
{C, F}	6
{D, E}	7
{B, E}	8

The Challenge: Detecting Cycles Efficiently

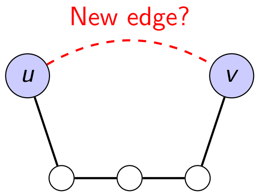
Bottleneck: How do we check if adding an edge creates a cycle?

Naive Approach (BFS / DFS):

- Traverse the graph starting from u.
- **Worst Case:** Must explore the entire connected component to find v (or prove it's unreachable).
- Cost per edge: $O(|V|)$
- **Total Cost:**

  $O(|E| \cdot |V|)$

  Too Slow for dense graphs!



Need to search entire component!

Notice: We don't need full path information!
We only need to know if u and v are in the same connected component.
This can be answered as a **Union-Find** problem much more efficiently!

Union-Find

Union-Find Data Structure

Goal: Maintain a partition of elements.

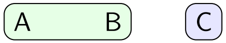
Required Operations

1. **make(x):** Create a new set containing only x
2. **find(x):** Return the *representative* of the set containing x
3. **union(x, y):** Merge the sets containing x and y

For Kruskal's:

- Each vertex starts in its own set
- `find(`u`) == find(`v`)` $\Longrightarrow$ same component
- If different: `union(`u`, `v`)` to merge them

Initial: 3 separate sets



After union(A, B): 2 sets

Array-Based Representation

Idea: Use an array `rep[]` where each element stores its component representative.

Structure:

- Array `rep[1..n]`
- `rep[v]` = representative of v's component
- Elements with same `rep` are in same component

Example:

	A	B	C	D	E
`rep[]`:	A	B	C	D	E

Initially: each element is its own representative

Operations:

- **find(v):** Return `rep[v]`
- **union(u, v):** Change all `rep`s matching one to match the other

	A	B	C	D	E
`rep[]`:	A	C	C	A	A

After `union(A, D)` and `union(D, E)`

Invariant: find(u) = find(v) $\iff$ u and v are in the same component

Naive Union-Find: Operation Costs

Implementation Idea: Array `rep[]` where `rep[v]` stores component's rep of v.

Individual Operation Complexities

- **find(v):** Return `rep[v]` $\Theta(1)$
- **union(u, v):** Scan entire array, change all entries matching `rep[u]` to `rep[v]` $\Theta(n)$

Why is union $\Theta(n)$?

- Must scan all n positions in the array
- Check if `rep[i]` matches the old component rep
- Update matching entries to new component rep
- Cannot avoid scanning the entire array — we don't know which entries need updating!

Bottleneck: Every union operation costs $\Theta(n)$ time $\implies$ $\Theta(n^2)$ in total.

Union: Improvement 1

Idea: Maintain a list of members for each component.

Additional Data Structure

members[r]: List of all vertices in the component with representative r

For example: If `rep[A]` = `rep[B]` = `rep[C]` = A, then:

$$\texttt{members[A]} = [A, B, C]$$

Invariant:

`members[rep[u]]` is a list of all nodes in the connected component of u

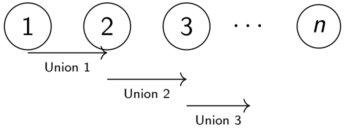
Runtime: $O(|\texttt{members}[r_u]|) = O(|\text{component of } u|)$
No longer need to scan all n positions!

Members List: Does This Solve Our Problem?

Question: Is $O(|\text{component of } u|)$ per union good enough?

Answer: Not quite... Consider the worst case:

Worst-case sequence:



Build a chain: union(1,2), union(2,3), ..., union($n-1, n$)

Cost Analysis:

***i*-th union call:**
After $i-1$ unions, we have a component of size i.
Union cost: $O(i)$

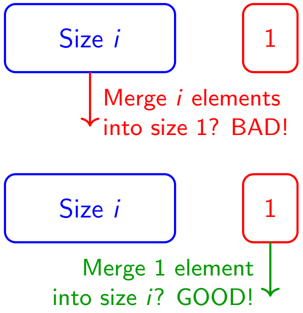
Total cost:

$$O(1+2+3+\cdots+(n-1))$$
$$= O\left(\frac{n(n-1)}{2}\right)$$
$$= O(n^2)$$

Problem: Still $O(n^2)$ total cost in the worst case...

Improvement 2: Weighted Union

Why did the worst case happen?
We kept merging the *larger* component into the *smaller* one!


Size i
1
Merge i elements into size 1? BAD!
Size i
1
Merge 1 element into size i? GOOD!

Weighted Union Rule: Always merge the **smaller** component into the **larger** one.
This ensures each element is moved at most $\log n$ times!

Weighted Union: Runtime

Observe: How many times can a single element's representative change?

Update Chain:

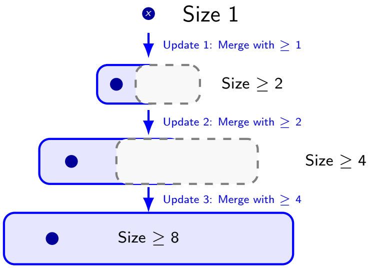
- Consider a specific element x.
- x's representative is only updated if x's set is the **smaller** one in a union.
- Therefore, after the union, the new set size is at least **double** the old size.

Component size: After k-th update: size $\geq 2^k$

Since max size is N:

$$2^k \leq n \implies k \leq \log_2 n$$

Visualized



Conclusion: Total complexity is $O(n \log n)$ because each of the n elements is moved at most $\log n$ times.

Union: Implementation

```
 1  void union(ArrayList<LinkedList> mems,
 2             int[] rep, int u, int v) {
 3
 4      // 1. OPTIMIZATION: Ensure u is smaller
 5      if (mems.get(u).size() > mems.get(v).size()) {
 6         int swap = u; u = v; v = swap;
 7      }
 8
 9      // 2. MERGE: Update smaller (u) into larger (v)
10      LinkedList<Integer> uList = mems.get(u);
11      LinkedList<Integer> vList = mems.get(v);
12
13      // Update representative for u itself
14      rep[u] = v;
15      vList.add(u);
16
17      // Update representatives for all members of u
18      for (Integer z : uList) {
19          rep[z] = v;      // The "Work"
20          vList.add(z);    // Move member
21      }
22  }
```

Note: Lines 6 and 18 guarantee we only touch the minority of elements.

Complexity Analysis

1. Single Union Cost

- Worst case: $O(n)$
- *(Happens if we merge two sets of size $n/2$)*

2. Aggregate Cost (Kruskal's)

- How many times does a *specific element's* `rep` change?
- Every time it changes, its set size **doubles**.
- Max set size is n.
- Therefore: $\leq \log_2 n$ updates per element.

Total Time: $O(n \log n)$

Kruskal: Pseudocode

```
function KRUSKAL($G = (V, E, c)$)
    $T \leftarrow \emptyset$                                        ▷ MST edges

    // Sort edges by weight
    Sort $E$ by increasing weight: $e_1, e_2, \dots, e_{|E|}$

    // Process edges in order
    for each edge $(u, v) \in E$ (in sorted order) do
        if find($u$) $\neq$ find($v$) then
            $T \leftarrow T \cup \{(u, v)\}$                        ▷ Add to MST
            union($u, v$)                                           ▷ Merge components
        end if
        if $|T| = |V| - 1$ then
            break                                                   ▷ MST complete
        end if
    end for
    return $T$
end function
```

Your Turn: Proving Kruskal's Correctness

Claim: Let A be the set of edges selected by Kruskal's. Then $A \subseteq T$ for some MST T.

Proof. Proceed by induction on $|A|$:

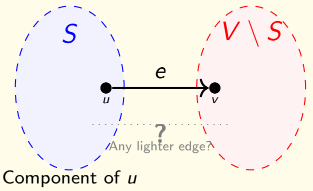
B.C.: $|A| =$ ______________________________

I.H.: Assume $A \subseteq T$ after selecting ____________ edges.

Inductive Step: Let $e = (u, v)$ be the next edge Kruskal adds and let S be the set of vertices in u's conn.-comp.

1. **Cut:** What can we say about e? *Does v belong to S?*
2. **Minimality:** We know Kruskal adds edges in ____________ order. (*Could there be an edge e' crossing $(S, V \setminus S)$ with $w(e') < w(e)$?*)
3. **Conclusion:** By the **Cut Property**:

Visualize the Step



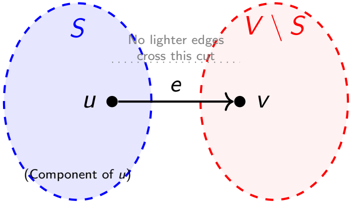
Kruskal: Correctness (1/2)

Kruskal's Invariant

Let A be the set of edges selected by Kruskal's algorithm. Then A is a subset of some Minimum Spanning Tree (MST).

Proof Strategy: Induction on $|A|$.

- **B.C.:** $A = \emptyset$ is trivially a subset of any MST.
- **I.H.:** Assume $A \subseteq T$ with $0 \leq |A| < n - 1$ for some MST T.



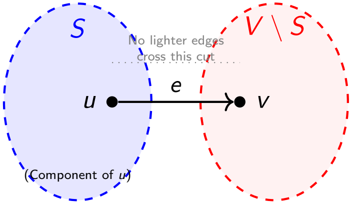
Lemma: If e is a minimum weight edge crossing any cut $(S, V \setminus S)$, then e belongs to some MST.

Kruskal: Correctness (2/2)

I.S.: Let $e = (u, v)$ be the next edge added and let S be the set of vertices in the component containing u.

1. Since adding e does not create a cycle, $v \notin S$. Thus, e crosses the cut $(S, V \setminus S)$.
2. Kruskal's scans edges by increasing weight. Any edge crossing $(S, V \setminus S)$ with weight $w(e') < w(e)$ would have been processed earlier.
3. Since u, v are currently disconnected, no such lighter edge exists.

   $\implies$ e is a **Light Edge** for cut $(S, V \setminus S)$
4. $A \cup \{e\}$ is also a subset of an MST.



Lemma: If e is a minimum weight edge crossing any cut $(S, V \setminus S)$, then e belongs to some MST.

In particular, this proves $A \subseteq T$ with $|A| = n - 1$ for some MST T. And since T has $n - 1$ edges, $A = T$. $\square$

Kruskal: Runtime

Kruskal's Complexity

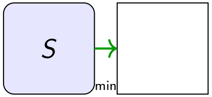
$$\begin{array}{rl} \text{Sorting edges:} & O(m \log m) = O(m \log n) \\ \text{Union-Find:} & O(m + n \log n) \\ \hline \textbf{Total:} & O(m \log n + m + n \log n) \\ & = O((m+n) \log n) \end{array}$$

Note: Since $m \leq n^2$, we have $\log m \leq 2 \log n$

Summary: Complete MST Picture

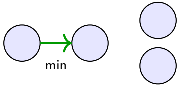
Three Greedy Algorithms, One Principle: The Cut Property

Prim's



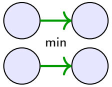
Grow one tree
Add min edge
leaving tree

Kruskal's



Sort all edges
Add min edge
not creating cycle

Boruvka's



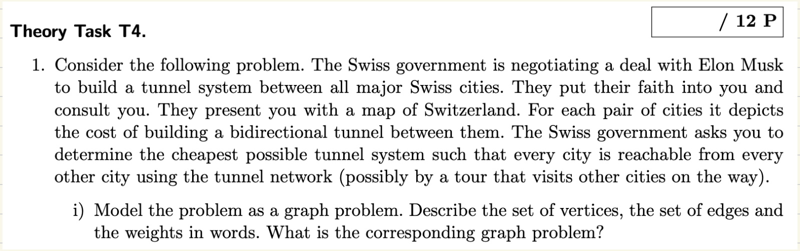
All components
add min edge
simultaneously

Key Takeaways:

- All three algorithms are correct by the Cut Property
- Runtime: $O(|V| + (|E|) \log |V|)$ for all three
- Kruskal's requires Union-Find for efficiency

Additional Practice

HS20



Theory Task T4. / 12 P

1. Consider the following problem. The Swiss government is negotiating a deal with Elon Musk to build a tunnel system between all major Swiss cities. They put their faith into you and consult you. They present you with a map of Switzerland. For each pair of cities it depicts the cost of building a bidirectional tunnel between them. The Swiss government asks you to determine the cheapest possible tunnel system such that every city is reachable from every other city using the tunnel network (possibly by a tour that visits other cities on the way).

   i) Model the problem as a graph problem. Describe the set of vertices, the set of edges and the weights in words. What is the corresponding graph problem?

HS20

ii) Use an algorithm from the lecture to solve the graph problem. State the name of the algorithm and its running time in terms of $|V|$ and $|E|$ in Θ-notation.

HS20

2. Now, the Swiss tunneling society contacts the government and proposes to build the tunnel between Basel and Geneva for half of Musk's cost. Thus, the government contacts you again. They want you to solve the following problem: Given the solution of the old problem in a) and an edge for which the cost is divided by two, design an algorithm that updates the solution such that the new edge cost is taken into account. *In order to achieve full points, your algorithm must run in time $\mathcal{O}(|V|)$.*

   Hint: You are only allowed to use the *solution* from 1., i.e. the set of tunnels in the chosen tunnel system. You are not allowed to use any intermediate computation results from your algorithm in 1.

   i) Describe your algorithm (for example, via pseudocode). A high-level description is enough.

HS20

ii) Prove the correctness of your algorithm and show that it runs in time $\mathcal{O}(|V|)$.

HS20

i) **Graph:** We model the problem as a weighted undirected graph $G = (V, E, c)$, where:

- the set of vertices V corresponds to the set of major Swiss cities
- the set of edges is $E = \{\{u, v\} \mid u, v \in V, u \neq v\}$
- the weight of an edge $c(e)$, $e = \{u, v\} \in E$, is given by the cost of construction of a tunnel between cities u and v.

Problem Definition: The solution to the given problem corresponds to finding an MST in G.

ii) **Runtime:**

We can find an MST of G using Kruskal in $O((|V| + |E|) \log(|V|))$.

HS20

i) Algorithm:

Let $\hat{e} = \{\text{Basel}, \text{Geneva}\}$ with new weight $c'(\hat{e}) = c(\hat{e})/2$.

1. **Case 1 ($\hat{e} \in T$):** Return $T' = T$.
2. **Case 2 ($\hat{e} \notin T$):**
   - Let P be the unique path between Basel and Geneva in T.
   - **Find** $e_{max} = \arg\max_{e \in P} c(e)$.
   - **Update:**

     If $c'(\hat{e}) < c(e_{max})$: Return $T' = (T \setminus \{e_{max}\}) \cup \{\hat{e}\}$.

     Else: Return $T' = T$.

Runtime: Finding path P and e_{max} takes $O(|V|)$ using DFS on the tree T (since $|E_T| = |V| - 1$). All other operations are $O(1)$. Total: $O(|V|)$.

HS20: Correctness (1/2)

Case 1: $\hat{e} \in T$

- Since $\hat{e} \in T$, reducing its weight decreases the total cost of T.
- Any other spanning tree T_{other} either contains $\hat{e}$ (cost decreases by same amount) or does not (cost stays same).
- Since T was optimal for c, it remains optimal for c' (c' being the new weight function).

(see case 2 on next slide)

HS20: Correctness (2/2)

Case 2: $\hat{e} \notin T$

- Removing e_{max} partitions V into two sets, S and $V \setminus S$. Let E_{cut} be the set of edges crossing $(S, V \setminus S)$ in the original graph.
- Since T is an MST for c, e_{max} is a minimum weight edge for this cut:
$$\forall e \in E_{cut}, \quad c(e_{max}) \leq c(e)$$
- For the new weights c', only $\hat{e}$ changes. Thus:
$$\forall e \in E_{cut} \setminus \{\hat{e}\}, \quad c'(e_{max}) \leq c'(e)$$
- The minimum weight edge for the cut in G is now $\min(e_{max}, \hat{e})$.
- If $c'(\hat{e}) < c'(e_{max})$, $\hat{e}$ is the unique minimum crossing edge and must be included (Cut Property). Otherwise, e_{max} remains minimal.
- Finally, any other edge in T cannot be replaced with a cheaper edge, since if we suppose to the contrary, then we could also improve the original tree given that no other edge weights have changed.

Hence, T' is a valid MST. $\square$

Your Turn: MST "Bottlenecks"

Define the *bottleneck capacity* of a path as the maximum weight of any edge on that path.

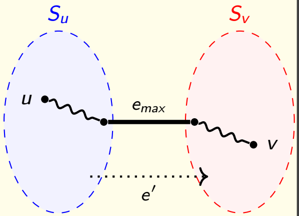
Claim: The path between any two nodes u, v in an MST minimizes the bottleneck capacity.
Guided Proof:

1. Let P_{MST} be the path in the MST between u and v. Let e_{max} be the heaviest edge in P_{MST}. Removing e_{max} creates a Cut (S_u, S_v) because ...

   What do we know about $w(e_{max})$ compared to other edges crossing this cut?

2. Assume for contradiction there is another path P_{alt} in G whose bottleneck is **strictly less** than $w(e_{max})$. Then:

Time: 5-8 Minutes



Does P_{alt} have to cross the gap?